ON THE DEGREE OF CONVERGENCE OF BIRKHOFF'S SERIES*

BY

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Introduction

In the linear differential equation

$$(1) \qquad \frac{d^n u}{dx^n} + P_2(x)\frac{d^{n-2} u}{dx^{n-2}} + \cdots + P_n(x)u + \lambda u = 0$$

let the coefficients $P_2(x), \cdots, P_n(x)$ be continuous with their derivatives of all orders in the closed interval (a, b), and let

$$(2) \qquad W_1(u) = 0, \qquad W_2(u) = 0, \qquad \cdots, \qquad W_n(u) = 0$$

be n linearly independent linear homogeneous conditions in $u(a)$, $u'(a)$, $\cdots$, $u^{(n-1)}(a)$; $u(b)$, $u'(b)$, $\cdots$, $u^{(n-1)}(b)$, which have been *normalized* and are *regular*.† Then it is known that the characteristic numbers of this system are in general simple, are infinite in number, and have no cluster point in the finite plane. If $U_1(x)$, $U_2(x)$, $U_3(x)$, $\cdots$ denote solutions of the system corresponding to simple characteristic numbers $\lambda_1, \lambda_2, \lambda_3, \cdots$, then an arbitrary function $f(x)$ may be represented, formally at least, by a series

$$(3) \qquad A_1 U_1(x) + A_2 U_2(x) + A_3 U_3(x) + \cdots .$$

The term $A_i U_i(x)$ is given by the equation

$$(4) \qquad A_i U_i(x) = \int_a^b f(t) R_i(x, t, \lambda)\, dt,$$

where $R_i(x, t, \lambda)$ is the residue at $\lambda = \lambda_i$ of the Green's function $G(x, t, \lambda)$ belonging to the system (1) and (2). If the characteristic numbers are not all simple we shall agree to define the terms of the series (3) by means of equation (4). Therefore in any case the sum of the first ν terms of (3) is

$$(5) \qquad S_\nu(x) = \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma_\nu} \int_a^b f(t)\, G(x, t, \lambda)\, dt d\lambda,$$

* Presented to the Society, December 28, 1915. The general nature of the series treated here was first discussed by Birkhoff, these Transactions, vol. 9 (1908), pp. 373–395. This paper is based on his results.

† For definitions of these terms see Birkhoff, loc. cit., p. 382.

where Γ_ν is a contour in the λ-plane inclosing the first ν characteristic numbers, and excluding all others.

It is the purpose of this paper to discuss the degree of convergence of the series (3) with suitable restrictions on the function $f(x)$. This is done by considering the order of magnitude of the remainder $f(x) - S_\nu(x)$. The methods are similar on the one hand to those used by Birkhoff in proving the convergence of (3), and on the other hand to those used by Jackson in finding the degree of convergence of Sturm-Liouville series.* It is found that if $f(x)$ with its first $m-1$ derivatives vanish at a and at b, the remainder after ν terms of the series is less than a constant multiple of $1/\nu^m$ when the mth derivative is of limited variation, and less than a constant multiple of $(\log \nu)/\nu^{m+1}$ when the mth derivative satisfies a Lipschitz condition.

1. Preliminary formulas

In this section the needed facts concerning the system (1) and (2) are briefly presented.

1. **The characteristic numbers.** The characteristic numbers of the system (1) and (2) form a pair of infinite sequences

$$\lambda_\nu' = -\left(\frac{2\nu\pi\sqrt{-1}}{b-a}\right)^n (1 + E'/\nu),$$

(6)

$$\lambda_\nu'' = -\left(\frac{-2\nu\pi\sqrt{-1}}{b-a}\right)^n (1 + E''/\nu) \qquad (\nu = 1, 2, 3, \cdots),$$

where E' and E'' are functions of ν that are bounded as ν becomes infinite.

2. **The Green's function.** The explicit form of the Green's function is

$$(7)\quad G(x,t,\lambda) = \frac{(-1)^n}{\Delta}\begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) & \bar{G}(x,t,\lambda) \\ W_1(y_1) & W_1(y_2) & \cdots & W_1(y_n) & W_1(\bar{G}) \\ W_2(y_1) & W_2(y_2) & \cdots & W_2(y_n) & W_2(\bar{G}) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ W_n(y_1) & W_n(y_2) & \cdots & W_n(y_n) & W_n(\bar{G}) \end{vmatrix},$$

where

$$(8)\qquad \Delta = \begin{vmatrix} W_1(y_1) & W_1(y_2) & \cdots & W_1(y_n) \\ W_2(y_1) & W_2(y_2) & \cdots & W_2(y_n) \\ \cdot & \cdot & \cdot & \cdot \\ W_n(y_1) & W_n(y_2) & \cdots & W_n(y_n) \end{vmatrix},$$

and

* These Transactions, vol. 15 (1914), pp. 439–466.

$$(9)\quad \bar{G}(x,t,\lambda) = \pm\frac{1}{2}\sum_{i=1}^{n} y_i(x)\,\bar{y}_i(t)$$

$$= \pm\frac{1}{2}\frac{\begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1^{(n-2)}(t) & y_2^{(n-2)}(t) & \cdots & y_n^{(n-2)}(t) \\ \cdot & \cdot & \cdot & \cdot \\ y_1(t) & y_2(t) & \cdots & y_n(t) \end{vmatrix}}{\begin{vmatrix} y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \\ y_1^{(n-2)}(t) & y_2^{(n-2)}(t) & \cdots & y_n^{(n-2)}(t) \\ \cdot & \cdot & \cdot & \cdot \\ y_1(t) & y_2(t) & \cdots & y_n(t) \end{vmatrix}}$$

$$(+ \text{ if } x > t, \ - \text{ if } x < t).$$

Here $y_1(x), y_2(x), \cdots, y_n(x)$ are any n linearly independent solutions of equation (1).

3. **Asymptotic form of solutions.** Let $\lambda = \rho^n$, let the ρ-plane be divided into $2n$ equal sectors

$$J_k: \qquad k\pi/n \leqq \arg\rho \leqq (k+1)\pi/n \quad (k = 0, 1, \cdots, n-1),$$

and let the n nth roots of -1, $w_1, w_2, \cdots, w_n$, be numbered in such a manner that when ρ is on any given sector J_k the inequalities

$$R(\rho w_1) \leqq R(\rho w_2) \leqq \cdots \leqq R(\rho w_n)$$

are satisfied, where $R(\rho w_i)$ denotes the real part of ρw_i. Then if the coefficients $P_s(x)$ in (1) have continuous derivatives of order $(m+n-s)$ in (a, b), m being any positive integer or zero, there exist, for every sector J_k, n independent solutions of (1) of the form

$$(10)\qquad y_i(x) = \bar{u}_i(x,\rho) + e^{\rho w_i(x-c_i)}\,E_{i0}/\rho^{m+1},$$

$$y_i^{(j)}(x) = \bar{u}_i^{(j)}(x,\rho) + e^{\rho w_i(x-c_i)}\,E_{ij}/\rho^{m-j+1} \quad (j = 1, 2, \cdots, n-1),$$

in which.

$$(11)\quad \bar{u}_i(x,\rho) = e^{\rho w_i(x-c_i)}[1 + \phi_1(x)/\rho w_i + \cdots + \phi_m(x)/(\rho w_i)^m].$$

The functions $\phi_j(x)$ are independent of i and ρ, and have continuous derivatives of order $(m+n-j)$ in (a, b), and the E_{ij} are analytic in ρ and bounded as ρ becomes infinite in J_k for all values of x in (a, b). Likewise the n functions $\bar{y}_i(t)$ of formula (9) have the form

$$(12)\quad \bar{y}_i(t) = \frac{e^{-\rho w_i(t-c_i)}}{n(\rho w_i)^{n-1}}[1 + \psi_1(t)/\rho w_i + \cdots + \psi_m(t)/(\rho w_i)^m + \bar{E}_i/\rho^{m+1}],$$

in which the $\psi_j(t)$ and the $\bar{E}_i$ have the same properties as the $\phi_j(x)$ and the E_{ij} respectively.*

4. **The constants c_i, and the functions v_i and u_i.** The constants c_i in each of the functions y_i and $\bar{y}_i$ may be chosen as follows:

$$c_i = a \text{ if } R(\rho w_i) < 0, \qquad c_i = b \text{ if } R(\rho w_i) > 0.$$

When n is odd and $i = (n+1)/2$, c_i has different values in the two halves of the sector J_k, but in all other cases c_i remains constant throughout the sector. By the given choice of the c_i we see that when x is in (a, b),

$$\left|e^{\rho w_i(x-c_i)}\right| \leqq 1 \qquad (i = 1, 2, \cdots, n).$$

Also if we let

$$c_i' = b \text{ if } R(\rho w_i) < 0, \quad \text{and} \quad c_i' = a \text{ if } R(\rho w_i) > 0,$$

then for t in (a, b),

$$\left|e^{-\rho w_i(t-c_i')}\right| \leqq 1 \qquad (i = 1, 2, \cdots, n).$$

It will be convenient to introduce the notation

$$v_i(t) = n(\rho w_i)^{n-1}\bar{y}_i(t) \qquad (i = 1, 2, \cdots, n), \tag{13}$$

and

$$u_i(t) = n(\rho w_i)^{n-1}e^{\pm\rho w_i(b-a)}\bar{y}_i(t) \tag{14}$$

$$(+ \text{ if } R(\rho w_i) < 0, \; - \text{ if } R(\rho w_i) > 0).$$

The explicit form of $u_i(t)$ is

$$u_i(t) = e^{-\rho w_i(t-c_i')}\left[1 + \frac{\psi_1(t)}{\rho w_i} + \cdots + \frac{\psi_m(t)}{(\rho w_i)^m} + \frac{\bar{E}_i}{\rho^{m+1}}\right]. \tag{15}$$

5. **The boundary conditions.** The normalized conditions (2) have the form†

$$W_i(u) = A_i(u) + B_i(u) = 0,$$

where

$$A_i(u) = \alpha_i u^{(k_i)}(a) + \sum_{j=0}^{k_i-1}\alpha_{ij}u^{(j)}(a),$$

$$B_i(u) = \beta_i u^{(k_i)}(b) + \sum_{j=0}^{k_i-1}\beta_{ij}u^{(j)}(b) \quad (k_1 \geqq k_2 \geqq \cdots \geqq k_n),$$

and no three successive k's are equal.

When the values of y_i as given in (10) are substituted for u in the above expressions it is found that

$$W_j(y_i) = \rho^{k_j}U_{ij} \quad (i = 1, 2, \cdots, n; j = 1, 2, \cdots, n), \tag{16}$$

* The formulas (10), (11), and (12) were first obtained by Birkhoff in a less explicit form, these Transactions, vol. 9 (1908), pp. 219–231. The forms used here are given in a note by the author in the Proceedings of the National Academy of Sciences, vol. 2 (1916), p. 543, and in the Bulletin of the American Mathematical Society, vol. 23 (1917), pp. 166–169.

† See second footnote on the first page of the article.

where U_{ij} is bounded as ρ becomes infinite on the given sector. Also when $R(\rho w_i) < 0$,

$$B_j(y_i) = \rho^{k_j} e^{\rho w_i(b-a)} V_{ij}, \tag{17}$$

and when $R(\rho w_i) > 0$,

$$A_j(y_i) = \rho^{k_j} e^{-\rho w_i(b-a)} V_{ij} \qquad (j = 1, 2, \cdots, n), \tag{18}$$

where V_{ij} is bounded on J_k as ρ becomes infinite.

6. **Continuity of derivatives of the coefficients.** Birkhoff assumed throughout his proof of convergence that the functions $P_s(x)$ had derivatives of all orders, but of this property he made no other use than to establish the existence of solutions like those in (10). Since such solutions exist if the functions $P_s(x)$ have continuous derivatives of order $m + n - s$, where m is any positive integer or zero, it is evident that all the properties of the system (1) and (2) that we have mentioned will hold with this modification of the hypotheses. Therefore it will no longer be assumed that $P_s(x)$ has derivatives of all orders.

2. Transformation of the contour integral

In this section the contour integral in (5) is put in a more serviceable form.

From (7), (9), and (13) it follows that after the change of variable $\lambda = \rho^n$, the contour integral can be put in the form

$$\sum_{i=1}^{n} \frac{(-1)^{n-1} w_i}{4\pi\sqrt{-1}} \int_{\gamma_\nu} \int_a^b \frac{f(t)}{\Delta} \cdot \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) & \pm y_i(x) \\ W_1(y_1) & W_1(y_2) & \cdots & W_1(y_n) & \overline{W}_1(y_i) \\ W_2(y_1) & W_2(y_2) & \cdots & W_2(y_n) & \overline{W}_2(y_i) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ W_n(y_1) & W_n(y_2) & \cdots & W_n(y_n) & \overline{W}_n(y_i) \end{vmatrix} v_i(t)\, dt\, d\rho \tag{19}$$

$$(+ \text{ if } x > t, \ - \text{ if } x < t),$$

where $\overline{W}_j(u) = -A_j(u) + B_j(u)$. The path of integration γ_ν is the segment $\gamma_{\nu,k}$ of a circle $|\rho| = r_\nu$ on the sector J_k, joined to the segment $\gamma_{\nu,k+1}$ of the same circle on the adjacent sector J_{k+1}.*

In (19) when $R(\rho w_i) > 0$ subtract the ith column from the last column, but when $R(\rho w_i) < 0$ add the ith column to the last column. Take the term in the upper right-hand corner out of the determinant and write it separately, putting zero in its place. Replace $W_j(y_i)$ and $A_j(y_i)$ and $B_j(y_i)$ (the last two of which will appear in the last column after the addition or subtraction) by their values from (16), (17), and (18). Cancel the factor

* When ρ is on J_k the functions y_i and $\bar{y}_i$ must be those belonging to J_k, and when ρ is on J_{k+1} they must be those belonging to J_{k+1}.

ρ^{k_j} from the $(j+1)$th row of the determinant in (19) with the same factor in the jth row of Δ. And finally remove the factor $e^{\pm\rho w_i(b-a)}$ ($+$ if $R(\rho w_i) < 0$, $-$ if $R(\rho w_i) > 0$) from the last column and combine it with $v_i(t)$, giving $u_i(t)$. When these steps have been taken it will be found that (19) takes the form

$$(20)\quad -\sum_{i=1}^{n}\frac{w_i}{2\pi\sqrt{-1}}\int_{\gamma_\nu}\int_{c_i}^{x} f(t)\,y_i(x)\,v_i(t)\,dt\,d\rho$$

$$-(-1)^n\sum_{i=1}^{n}\frac{w_i}{2\pi\sqrt{-1}}\int_{\gamma_\nu}\frac{\begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) & 0 \\ U_{11} & U_{12} & \cdots & U_{1n} & V_{1i} \\ U_{21} & U_{22} & \cdots & U_{2n} & V_{2i} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ U_{n1} & U_{n2} & \cdots & U_{nn} & V_{ni} \end{vmatrix}}{\begin{vmatrix} U_{11} & U_{12} & \cdots & U_{1n} \\ U_{21} & U_{22} & \cdots & U_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ U_{n1} & U_{n2} & \cdots & U_{nn} \end{vmatrix}}\cdot\left[\int_a^b f(t)\,u_i(t)\,dt\right]d\rho .$$

The sequence of paths of integration γ_ν may be so chosen that the distance from the nearest ρ_i corresponding to a characteristic number λ_i is uniformly greater than some constant δ. Then when ρ is on any path γ_ν the determinant in the denominator of (20) remains uniformly greater in absolute value than some positive constant d. It is therefore evident that after the determinant in the numerator is expanded by minors of the top row, the second sum in (20) may be written

$$(21)\qquad R_\nu(x) = \sum_{i=1}^{n}\sum_{j=1}^{n}\int_{\gamma_\nu} E_{ij}\,y_j(x)\left[\int_a^b f(t)\,u_i(t)\,dt\right]d\rho,$$

where the E_{ij} are bounded as ρ becomes infinite along the sequence of paths γ_ν.

From (10), (12), and (13) it follows that*

$$(22)\qquad y_i(x)\,v_i(t) = e^{\rho w_i(x-t)}\left[1+\sum_{j=1}^{m}\omega_j(x,t)/(\rho w_i)^j + E_i/\rho^{m+1}\right],$$

where $\omega_j(x,t)$ is equal to

$$\sum_{k=1}^{j-1}\phi_k(x)\,\psi_{j-k}(t),$$

and therefore has a continuous $(m+n-j)$th derivative with respect to t in (a,b).

* The letter E will regularly be used to denote functions of ρ and x that are bounded on the sector to which they belong as ρ becomes infinite.

Let

$$\xi(x, t, \rho w_i) = e^{\rho w_i(x-t)} \sum_{j=1}^{m} \omega_j(x, t)/(\rho w_i)^j .$$

Then if we make the transformation $\rho w_i = z$ in the ith term of the sum

$$\frac{-1}{2\pi\sqrt{-1}} \sum_{i=1}^{n} \int_{\gamma_\nu} \int_{c_i}^{x} f(t)\xi(x, t, \rho w_i)\, dt\, w_i\, d\rho \quad (i = 1, 2, \cdots, n),$$

and combine the new paths of integration, the result may be written

$$L_\nu(x) = \frac{-1}{2\pi\sqrt{-1}} \left[\int_{l_1} \int_{a}^{x} f(t)\xi(x, t, z)\, dt\, dz - \int_{l_2} \int_{x}^{b} f(t)\xi(x, t, z)\, dt\, dz \right], \tag{23}$$

where l_1 is the semicircle of radius $|z| = r_\nu$ in the left half of the z-plane, and l_2 is the semicircle in the right-half plane.*

Similarly

$$\frac{-1}{2\pi\sqrt{-1}} \sum_{i=1}^{n} \int_{\gamma_\nu} \int_{c_i}^{x} f(t)\, e^{\rho w_i(x-t)}\, dt\, w_i\, d\rho = \frac{-1}{2\pi\sqrt{-1}} \left[\int_{l_1} \int_{a}^{x} f(t)\, e^{z(x-t)}\, dt\, dz - \int_{l_2} \int_{x}^{b} f(t)\, e^{z(x-t)}\, dt\, dz \right]. \tag{24}$$

The right-hand side of this last equation we can simplify further by changing the order of integration and noting that

$$-\int_{l_1} e^{z(x-t)}\, dz = \int_{l_2} e^{z(x-t)}\, dz .$$

For then the right-hand side of (24) becomes

$$\frac{1}{2\pi\sqrt{-1}} \int_{a}^{b} f(t) \int_{-\sqrt{-1}r_\nu}^{\sqrt{-1}r_\nu} e^{z(x-t)}\, dz\, dt,$$

which upon integration with respect to z becomes

$$\frac{1}{\pi} \int_{a}^{b} f(t) \frac{\sin r_\nu (x - t)}{x - t}\, dt.$$

It remains to treat the terms arising from the integration of the E terms in (22). We restrict $f(x)$ to be bounded in (a, b), and let $M = \max|f(x)|$ in (a, b). Since E_i is bounded as ρ becomes infinite along the sequence of paths γ_ν for all values of x and t in (a, b), we may let $K = \max|E_i|$ $(i = 1, 2, \cdots, n)$. Now set $\rho w_i = r_\nu(\sin\phi + \sqrt{-1}\cos\phi)$, let g_i be the arc on

* A similar transformation is used by Tamarkine, Rendiconti del Circolo Matematico di Palermo, vol. 34 (1912), pp. 345–382.

which ρw_i lies when ρ is on γ_ν, and let ϕ_1 and ϕ_2 be the angles made with the axis of imaginaries by radii drawn to the extremities of g_i. Let $|x - c_i| = h$. Then

$$\left| \int_{\gamma_\nu} \int_{c_i}^{x} f(t) \frac{e^{\rho w_i(x-t)} E_i}{\rho^{m+1}} dt\, w_i\, d\rho \right|$$
$$\leqq \frac{MK}{r_\nu^m} \left| \int_{\phi_1}^{\phi_2} \int_{c_i}^{x} e^{r_\nu \sin \phi (x-t)} dt\, d\phi \right| = \frac{MK}{r_\nu^{m+1}} \left| \int_{\phi_1}^{\phi_2} \frac{1 - e^{-r_\nu h \sin \phi}}{\sin \phi} d\phi \right|$$
$$\leqq \frac{2MK}{r_\nu^{m+1}} \int_0^{\pi/2} \frac{1 - e^{-r_\nu h \sin \phi}}{\sin \phi} d\phi < \frac{2MK}{r_\nu^{m+1}} \left[hr_\nu \int_0^{1/r_\nu} d\phi + \int_{1/r_\nu}^{\pi/2} \frac{\pi d\phi}{2\phi} \right]$$
$$= \frac{2MK}{r_\nu^{m+1}} \left[h + \frac{\pi}{2} \log (\pi r_\nu / 2) \right] = M \frac{E_\nu \log r_\nu}{r_\nu^{m+1}},$$

where E_ν is independent of $f(x)$ and is bounded as r_ν becomes infinite. To verify the last inequality we have only to note that

$$\frac{1 - e^{-r_\nu h \sin \phi}}{\sin \phi}$$

is at most equal to $r_\nu h$ in the interval $0 \leqq \phi \leqq 1/r_\nu$, and is less than $\pi/(2\phi)$ in the interval $1/r_\nu \leqq \phi \leqq \pi/2$.

Let $\phi(\nu)$ and $\psi(\nu)$ be any two real functions of ν. Then if $\phi(\nu)/\psi(\nu)$ is bounded as ν becomes infinite, that fact will be indicated by the notation

$$\phi(\nu) = O(\psi(\nu)).$$

In this notation

$$\frac{E_\nu \log r_\nu}{r_\nu^{m+1}} = O\left(\frac{\log \nu}{\nu^{m+1}} \right),$$

as is readily seen from (6), since $2r_\nu = \sqrt[n]{|\lambda_\nu|} + \eta_\nu$, where η_ν may be taken less than some constant independent of ν.

Collecting the results obtained so far, we write the sum of ν terms of the series as follows:

$$S_\nu(x) = \frac{1}{\pi} \int_a^b f(t) \frac{\sin r_\nu (x-t)}{x-t} dt + L_\nu(x) + R_\nu(x) + O\left(\frac{\log \nu}{\nu^{m+1}} \right). \tag{25}$$

The constant implied in the O-notation is independent of x, so the equation (25) holds *uniformly* with respect to x.

3. Order of magnitude of the terms $L_\nu(x)$ and $R_\nu(x)$

In this section the terms $L_\nu(x)$ and $R_\nu(x)$ are evaluated for a restricted class of functions $f(x)$. The conclusions are given in Lemmas 2 and 3, for the proof of which the following lemma is needed.

Lemma 1. *If $F(x)$ is any continuous function of limited variation in (a, b),* *then*

$$\int_a^b F(t)\, e^{-\rho w_i(t-c_i')}\, dt = E/r_\nu \qquad (i = 1, 2, \cdots, n),$$

where as usual $r_\nu = |\rho|$.

Since $F(t)$ is continuous and of limited variation in (a, b), there exist two continuous functions F_1 and F_2, each positive and monotone increasing in (a, b), and such that $F = F_1 - F_2$. Write $\rho w_i = r_\nu(p + \sqrt{-1}q)$, where p and q are real. Then in the case where $R(\rho w_i) < 0$ and where consequently $c_i' = b$, we have

$$\int_a^b F(t)\, e^{-\rho w_i(t-b)}\, dt$$
$$= \sum_{j=1}^{2} (-1)^{j-1} \int_a^b F_j(t)\, e^{-r_\nu p(t-b)} [\cos r_\nu q(t-b) - \sqrt{-1} \sin r_\nu q(t-b)]\, dt.$$

Consider the integral

$$\int_a^b F_1(t)\, e^{-r_\nu p(t-b)} \cos r_\nu q(t-b)\, dt.$$

By the second law of the mean it is equal to

$$F_1(b) \int_{\bar{a}}^b e^{-r_\nu p(t-b)} \cos r_\nu q(t-b)\, dt \qquad (a < \bar{a} < b),$$

which after integration becomes

$$\frac{F_1(b)}{r_\nu} [e^{r_\nu p(b-\bar{a})} [q \sin r_\nu q(b - \bar{a}) + p \cos r_\nu q(b - \bar{a})] - p].$$

This expression is less in absolute value than $3F_1(b)/r_\nu$, since p is negative. Similar arguments apply to the three remaining integrals, and to the case where $R(\rho w_i) > 0$.

Lemma 2. *Let the function* $f(x)$ *have a continuous mth derivative of limited variation in the interval* (a, b), *and let* $f(x)$ *and its first* $m - 1$ *derivatives vanish at a and at* b. *Then*

I. $R_\nu(x) = O(1/\nu^m)$ *uniformly in* (a, b), *and*

II. $R_\nu(x) = \frac{1}{h} O(1/\nu^{m+1})$ *uniformly in the interval* (a', b'), *where* $a' - a = h$, *and* $b - b' = h$.

First consider the integral

$$\int_a^b f(t)\, u_i(t)\, dt,$$

which by (15) is equal to

$$\int_a^b f(t)e^{-\rho w_i(t-c_i')}\,dt + \sum_{j=1}^{m}\frac{1}{(\rho w_i)^j}\int_a^b f(t)\psi_j(t)e^{-\rho w_i(t-c_i')}\,dt + \frac{1}{\rho^{m+1}}\int_a^b f(t)\,\bar{E}_i\,e^{-\rho w_i(t-c_i')}\,dt. \tag{26}$$

In the first integral of (26) integrate by parts m times. Since $f(t)$ and its first $m-1$ derivatives vanish at both limits of integration, all the integrated terms disappear, giving

$$\int_a^b f(t)e^{-\rho w_i(t-c_i')}\,dt = \frac{1}{(\rho w_i)^m}\int_a^b f^{(n)}(t)e^{-\rho w_i(t-c_i')}\,dt,$$

where $f^{(m)}(t)$ satisfies the hypotheses of Lemma 1, so that the right hand side is E/r_ν^{m+1}. Similarly after integration by parts $m+1-j$ times we have

$$\frac{1}{(\rho w_i)^j}\int_a^b f(t)\psi_j(t)e^{-\rho w_i(t-c_i')}\,dt = \frac{1}{(\rho w_i)^{m+1}}\int_a^b \frac{d^{m+1-j}}{dt^{m+1-j}}[f(t)\psi_j(t)]\,e^{-\rho w_i(t-c_i')}\,dt.$$

Now $\psi_j(t)$ has a continuous $(m+n-j)$th and therefore at least a continuous $(m+1-j)$th derivative in (a,b), so that

$$\frac{d^{m+1-j}}{dt^{m+1-j}}[f(t)\psi_j(t)]$$

must be continuous and therefore bounded in (a,b). Hence if we take absolute values we readily find that the right-hand term of the last equation is of the form E/r_ν^{m+1}. Similar arguments apply to the last term in (26). Therefore

$$\int_a^b f(t)u_i(t)\,dt = E/r_\nu^{m+1} \qquad (i=1,2,\cdots,n). \tag{27}$$

Now substitute from (27) into (21) and take absolute values. Since $y_i(x)$ is bounded in (a,b) as ρ becomes infinite we have

$$|R_\nu(x)| = \frac{E}{r_\nu^{m+1}}\int_{\gamma_\nu}|d\rho| = E/r_\nu^m = O(1/\nu^m).$$

This proves I.

To prove II write

$$y_j(x) = e^{\rho w_j(x-c_j)}\,E_j,$$

a form that is obviously possible because of (10). Then by (27)

$$|R_\nu(x)| \leqq \sum_{j=1}^{m}\frac{E}{r_\nu^{m+1}}\int_{\gamma_\nu}|e^{\rho w_j(x-c_j)}|\,|w_j\,d\rho|.$$

Let h denote the smaller of the two quantities $|x-a|$ and $|x-b|$. Then in the notation used previously

$$|R_\nu(x)| \leqq \frac{nE}{r_\nu^m}\left|\int_{\phi_1}^{\phi_2} e^{-r_\nu h \sin\phi}\, d\phi\right|$$
$$\leqq \frac{E'}{r_\nu^m}\left|\int_0^{\pi/2} e^{-2r_\nu h\phi/\pi}\, d\phi\right|$$
$$\leqq \frac{E''}{hr_\nu^{m+1}} = \frac{1}{h}O(1/\nu^{m+1}).$$

LEMMA 3. *If the hypotheses of Lemma 2 are satisfied then*

$$L_\nu(x) = O(1/\nu^{m+1})$$

uniformly in (a, b).

Let

$$F_{jk}(x, t) = \frac{d^k}{dt^k}[f(t)\,\omega_j(x, t)] \quad (k = 0, 1, \cdots, m - j + 1).$$

Then $F_{jk}(x, t)$ vanishes at $t = a$ and at $t = b$ for $k = 0, 1, \cdots, m - j$. Accordingly after integration by parts $(m - j + 1)$ times

$$\frac{1}{z^j}\int_a^x F_{j0}(x, t)\, e^{-z(t-x)}\, dt$$
$$= -\sum_{k=0}^{m-j}\frac{F_{jk}(x, x)}{z^{k+j+1}} + \frac{1}{z^{m+1}}\int_a^x F_{j,\,m-j+1}(x, t)\, e^{-z(t-x)}\, dt.$$

Now $F_{j,\,m-j+1}(x, t)$ is of limited variation with respect to t in (a, b) and therefore an easy extension of Lemma 1 shows that the second term on the right in the last equation is E/r_ν^{m+2}. In a similar manner

$$\frac{1}{z^j}\int_x^b F_{j0}(x, t)\, e^{-z(t-x)}\, dt = \sum_{k=0}^{m-j}\frac{F_{jk}(x, x)}{z^{k+j+1}} + E/r_\nu^{m+2}.$$

Therefore

$$-\int_a^x f(t)\,\xi(x, t, z)\, dt = \sum_{j=1}^{m}\sum_{k=0}^{m-j}\frac{F_{jk}(x, x)}{z^{k+j+1}} + E/r_\nu^{m+2},$$
$$\int_x^b f(t)\,\xi(x, t, z)\, dt = \sum_{j=1}^{m}\sum_{k=0}^{m-j}\frac{F_{jk}(x, x)}{z^{k+j+1}} + E/r_\nu^{m+2}.$$

When these expressions are put into (23), and when the two paths of integration are combined into the single circle l, the result is

$$\frac{1}{2\pi\sqrt{-1}}\sum_{j=1}^{m}\sum_{k=0}^{m-j} F_{jk}(x, x)\int_l \frac{dz}{z^{k+j+1}} + E/r_\nu^{m+1} = L_\nu(x).$$

The integral around the closed contour is zero for every pair of values (k, j) in the summation, since $k + j + 1$ is always greater than 1; and therefore

$$L_\nu(x) = E/r_\nu^{m+1} = O(1/\nu^{m+1}).$$

Note. It was solely for the purpose of obtaining this result that it was found necessary to employ more precise asymptotic forms for the solutions $y_i(x)$ and for the $\bar{y}_i(t)$ than those used by Birkhoff. It was only because of the fact that the functions $\phi_j(x)$ and $\psi_j(x)$ were *independent of i* that $L_\nu(x)$ could be put in the form (23), from which the conclusion of Lemma 3 was derived.

4. Degree of convergence of the series

The results obtained in the preceding sections enable us now to draw certain conclusions concerning the degree of convergence of the series.

Theorem 1. *Let $f(x)$ have a continuous mth derivative of limited variation in (a, b), and let $f(x)$ and its first $m-1$ derivatives vanish at a and at b. Let the coefficients $P_s(x)$ in the differential equation have continuous derivatives of order $(m+n-s)$. Then*

$$S_\nu(x) = f(x) + O(1/\nu^m)$$

uniformly in (a, b).

Under the hypotheses of the theorem we obtain from (25), by Lemma 2 I and Lemma 3,

$$S_\nu(x) = \frac{1}{\pi}\int_a^b f(t)\frac{\sin r_\nu(x-t)}{x-t}\,dt + O(1/\nu^m),$$

uniformly in (a, b).

Now the particular differential system which yields the Fourier's series is the following:

$$\frac{d^2 u}{dx^2} + \lambda u = 0, \tag{28}$$

$$u(a) = u(b), \qquad u'(a) = u'(b),$$

a system that evidently satisfies all the conditions that we have imposed on the system (1) and (2). Moreover, although the characteristic numbers of this system are not simple, it is easily shown by direct computation that the contour integral corresponding to that in (5) formed for the system (28) actually represents the sum of the first ν terms of the Fourier's series. In this case in counting the number of characteristic numbers inclosed in the contour it must be remembered that each one counts double.

From these considerations it follows that if $\Sigma_\nu(x)$ denotes the sum of the first ν terms of the Fourier's series for $f(x)$, then

$$\Sigma_\nu(x) = \frac{1}{\pi}\int_a^b f(t)\frac{\sin \bar{r}_\nu(x-t)}{x-t}\,dt + O(1/\nu^m).$$

Now when ν is sufficiently large, $\nu > \mu$, we can certainly choose the radii of the paths of integration in the ρ-plane the same for both systems, *i. e.*, make $r_\nu = \bar{r}_\nu$, for $\nu = \mu, \mu+1, \mu+2, \cdots$. For we have

$$\bar{r}_\nu = \frac{\pi\nu}{b-a} + q',$$

and

$$r_\nu = \frac{\pi\nu}{b-a} + q'',$$

where q' is chosen to keep the paths $\overline{\gamma}_\nu$ for the Fourier's series uniformly away from the characteristic numbers of (28), while q'' is chosen to keep the paths γ_ν uniformly away from the numbers ρ_ν. From the distribution of the characteristic numbers for the two systems we infer that when ν is large we may choose $q' = q''$. Then

$$S_\nu(x) - \Sigma_\nu(x) = O(1/\nu^m).$$

But when $f(x)$ satisfies the hypotheses of the theorem it is known that

$$\Sigma_\nu(x) = f(x) + O(1/\nu^m).$$

Therefore

$$S_\nu(x) = f(x) + O(1/\nu^m).$$

THEOREM 2. *Let $f(x)$ have an mth derivative satisfying the Lipschitz condition*

$$|f^{(m)}(x_2) - f^{(m)}(x_1)| \leqq N|x_2 - x_1|$$

in (a, b), and let $f(x)$ and its first $m-1$ derivatives vanish at a and at b. Let the coefficients $P_s(x)$ have continuous derivatives of order $(m+n-s)$ in (a, b). Then

$$S_\nu(x) = f(x) + O\left(\frac{\log \nu}{\nu^{m+1}}\right)$$

uniformly in (a', b'), where $a' - a = b - b' = O(1/\log \nu)$.

Any function satisfying a Lipschitz condition will also be continuous and of limited variation. Therefore Lemmas 2 II and 3 hold for $f(x)$ with the new hypotheses, and consequently

$$L_\nu(x) = O(1/\nu^{m+1}),$$

and

$$R_\nu(x) = \frac{1}{h} O(1/\nu^{m+1}) = O\left(\frac{\log \nu}{\nu^{m+1}}\right),$$

provided that

$$h = O(1/\log \nu).$$

Hence

$$S_\nu(x) - \Sigma_\nu(x) = O\left(\frac{\log \nu}{\nu^{m+1}}\right).$$

But with the given hypotheses it is known* that

$$\Sigma_\nu(x) = f(x) + O\left(\frac{\log \nu}{\nu^{m+1}}\right).$$

* D. Jackson, these Transactions, vol. 13 (1912), pp. 491–515, Theorem V, and vol. 14 (1913), pp. 343–366, Theorem X.

Therefore

$$S_\nu(x) = f(x) + O\left(\frac{\log \nu}{\nu^{m+1}}\right)$$

uniformly in (a', b'), where $a' - a = b - b' = h = O(1/\log \nu)$.

In Theorem 2 nothing is said about the constant implied in the O-notation. A result which is more precise in this respect is as follows:

THEOREM 3. *Let $f(x)$ satisfy the hypotheses of Theorem 2, but let the coefficients $P_s(x)$ have continuous derivatives of order $(m + n - s + 1)$ in (a, b). Then for x in (a', b') and for ν sufficiently large*

$$|S_\nu(x) - f(x)| \leqq C_m N \frac{\log \nu}{\nu^{m+1}},$$

where C_m is a constant depending only on m. Here $a' - a = b - b' = h > 0$, where h is independent of ν.

With the hypotheses of the present theorem the asymptotic formulas for y_i and $\bar{y}_i$ can be carried out to terms in $1/\rho^{m+2}$. It will therefore follow from the reductions of § 2 that

$$S_\nu(x) = \frac{1}{\pi}\int_a^b f(t)\frac{\sin r_\nu(x-t)}{x-t}\,dt + L_\nu(x) + R_\nu(x) + O\left(\frac{\log \nu}{\nu^{m+2}}\right).$$

From Lemma 2 II and Lemma 3 (both of which hold of course under the present hypotheses), we have

$$S_\nu(x) = \frac{1}{\pi}\int_a^b f(t)\frac{\sin r_\nu(x-t)}{x-t}\,dt + O(1/\nu^{m+1}),$$

when x is in (a', b') interior to (a, b). Therefore

$$S_\nu(x) - \Sigma_\nu(x) = O(1/\nu^{m+1}).$$

But with the hypotheses of the theorem it is known* that

$$|\Sigma_\nu(x) - f(x)| \leqq \tfrac{1}{2} C_m N \frac{\log \nu}{\nu^{m+1}},$$

where C_m depends only on m. Therefore when ν is large enough

$$|S_\nu(x) - f(x)| \leqq C_m N \frac{\log \nu}{\nu^{m+1}},$$

for x in (a', b').

BOWDOIN COLLEGE,
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* See next preceding footnote.